A proof of simultaneous linearization with a polylog estimate

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February 2, 2008

Abstract

We give an alternative proof of simultaneous linearization recently shown by T.Ueda, which connects the Schröder equation and the Abel equation analytically. Indeed, we generalize Ueda's original result so that we may apply it to the parabolic fixed points with multiple petals. As an application, we show a continuity result on linearizing coordinates in complex dynamics.

1 Introduction

Let us start with a work out example to explain a motivation to consider the simultaneous linearization theorem.

Cauliflowers. In the family of quadratic maps, the simplest parabolic fixed point is given by $g(w) = w + w^2$. (Whose Julia set is called the *cauliflower*.) Now we consider its perturbation of the form $f(w) = \lambda w + w^2$ with $\lambda \nearrow 1$. According to [Mi, §8 and §10], we have the following fact:

Proposition 1.1 (Königs and Fatou coordinates) *Let K_f and K_g be the filled Julia sets of f and g. Then we have the following:*

1. *There exists a unique holomorphic branched covering map $\phi_f : K_f^\circ \to \mathbb{C}$ satisfying the Schröder equation $\phi_f(f(w)) = \lambda \phi_f(w)$ and $\phi_f(0) = \phi_f(-\lambda/2) - 1 = 0$. ϕ_f is univalent near $w = 0$.*

2. *There exists a unique holomorphic branched covering map $\phi_g : K_g^\circ \to \mathbb{C}$ satisfying the Abel equation $\phi_g(g(w)) = \phi_g(w) + 1$ and $\phi_g(-1/2) = 0$. ϕ_g is univalent on a disk $|w + r| < r$ with small $r > 0$.*

Note that $-\lambda/2$ and $-1/2$ are the critical points of f and g respectively.

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Observation. Set $\tilde{w} = \phi_f(w)$. Now the proposition above claims that the action of $f|_{K_f^\circ}$ is semiconjugated to $\tilde{w} \mapsto \lambda\tilde{w}$ by ϕ_f. Let us consider a Möbius map $W = S_f(\tilde{w}) = \lambda(\tilde{w}-1)/(\lambda-1)\tilde{w}$ that sends $\{0,1,\lambda\}$ to $\{\infty,0,1\}$ respectively. By taking conjugation by S_f, the action of $\tilde{w} \mapsto \lambda\tilde{w}$ is viewed as $W \mapsto W/\lambda + 1$. Let us set $W = \Phi_f(w) := S_f \circ \phi_f(w)$. Now we have

$$\Phi_f(f(w)) \;=\; \Phi_f(w)/\lambda + 1 \quad \text{and} \quad \Phi_f(-\lambda/2) \;=\; 0.$$

in total. On the other hand, by setting $W = \Phi_g(w) := \phi_g(w)$, we can see the action of $g|_{K_g^\circ}$ as $W \mapsto W + 1$. Thus we have

$$\Phi_g(g(w)) \;=\; \Phi_g(w) + 1 \quad \text{and} \quad \Phi_g(-1/2) \;=\; 0.$$

If λ tends to 1, that is, $f \to g$, the semiconjugated action in W-coordinate converges uniformly on compact sets. Now it would be natural if Φ_f tends to Φ_g. However, as one can see by referring the proof of the proposition in [Mi, §8 and §10], ϕ_f and ϕ_g are given in completely different ways thus we cannot conclude the convergence of $\Phi_f \to \Phi_g$ a priori.

Figure 1: Semiconjugation inside the filled Julia sets of Cauliflowers.

But there is another evident that supports this observation. Figure 1 shows the equipotential curves of ϕ_f and ϕ_g in the filled Julia sets. We can find similar patterns and it seems one converges to another. Actually, we have the following:

Theorem 1.2 *For any compact set $E \subset K_g^\circ$,*

(1) $E \subset K_f^\circ$ for all $f \approx g$; and

(2) $\Phi_f \to \Phi_g$ uniformly on E as $f \to g$.

Here $f \approx g$ means that f is sufficiently close to g, equivalently, λ sufficiently close to 1. (See [Ka, Theorem 5.5] for more generalized version of this proposition.) The proof of this theorem is given in Section 5, by using the *simultaneous linearization theorem.*

2   Simultaneous linearization

In this section we state the simultaneous linearization theorem. We first generalize the cauliflower setting above:

Perturbation of parabolics. Let f be an analytic map defined on a neighborhood of 0 in $\bar{\mathbb{C}}$ which is tangent to identity at 0. That is, f near 0 is of the form

$$f(w) \;=\; w + Aw^{m+1} + O(w^{m+2})$$

where $A \neq 0$ and $m \in \mathbb{N}$. By taking a linear coordinate change $w \mapsto A^{1/m}w$, we may assume that $A = 1$. In the theory of complex dynamics such a germ appears when we consider iteration of local dynamics near the parabolic periodic points, and plays very important roles. (See [Mi, §10] and [Sh] for example.) Now we consider a perturbation $f_\epsilon \to f$ of the form

$$f_\epsilon(w) \;=\; \Lambda_\epsilon w \left(1 + w^m + O(w^{m+1})\right)$$

with $\Lambda_\epsilon \to 1$ as $\epsilon \to 0$. By taking branched coordinate changes $z = -\Lambda_\epsilon^m/(mw^m)$ and setting $\tau_\epsilon := \Lambda_\epsilon^{-m}$, we have

$$f_\epsilon(z) \;=\; \tau_\epsilon z + 1 + O(|z|^{-1/m})$$
$$\longrightarrow f_0(z) \;=\; z + 1 + O(|z|^{-1/m})$$

uniformly near $w = \infty$ on the Riemann sphere $\hat{\mathbb{C}}$. The simultaneous linearization theorem will give partially linearizing coordinates of f_ϵ that depend continuously on ϵ when $\tau_\epsilon \to 1$ non-tangentially to the unit circle.

Let us formalize non-tangential accesses to 1 in the complex plane: After C.McMullen, for a variable $\tau \in \mathbb{C}$ converging to 1, we say $\tau \to 1$ *radially* (or more precisely, α-*radially*) if τ satisfies $|\arg(\tau - 1)| \leq \alpha$ for some fixed $\alpha \in [0, \pi/2)$.

Ueda's modulus. Let us consider a continuous family of complex numbers $\{\tau_\epsilon\}$ with $\epsilon \in [0,1]$ such that $|\tau_\epsilon| \geq 1$ and $\tau_\epsilon \to 1$ α-radially as $\epsilon \to 0$. For simplicity we assume that $\tau_\epsilon = 1$ iff $\epsilon = 0$. Set $\ell_\epsilon(z) := \tau_\epsilon z + 1$, which is an isomorphism of the Riemann sphere $\hat{\mathbb{C}}$. If $\epsilon > 0$, then $b_\epsilon := 1/(1 - \tau_\epsilon)$ is the repelling fixed point of ℓ_ϵ with $\ell_\epsilon(z) - b_\epsilon = \tau_\epsilon(z - b_\epsilon)$. Thus the function

$$N_\epsilon(z) \;:=\; |z - b_\epsilon| - |b_\epsilon|$$

satisfies the uniformly increasing property

$$N(\ell_\epsilon(z)) \;=\; |\tau_\epsilon|N(z) + \frac{|\tau_\epsilon| - 1}{|\tau_\epsilon - 1|} \;\geq\; |\tau_\epsilon|N(z) + \cos\alpha.$$

Similarly, if $\epsilon = 0$, the function

$$N_0(z) \;:=\; \sup\left\{\mathrm{Re}\,(e^{i\theta}z) \;:\; |\theta| < \alpha\right\}$$

also has the corresponding property

$$N_0(\ell_0(z)) \;\geq\; N_0(z) + \cos\alpha.$$

In both cases, set

$$\mathbb{V}_\epsilon(R) \ := \ \{z \in \mathbb{C} : N_\epsilon(z) \geq R\}$$

for $R > 0$. One can check that $N_\epsilon(z) \leq |z|$ and $\mathbb{V}_\epsilon(R) \subset \mathbb{B}(R) := \{z \in \mathbb{C} : |z| \geq R\}$ for all $\epsilon \in [0,1]$.

We will establish:

Theorem 2.1 (Simultaneous linearization) *Let* $\{f_\epsilon : \epsilon \in [0,1]\}$ *be a family of holomorphic maps on* $\mathbb{B}(R)$ *such that as* $\epsilon \to 0$ *we have the uniform convergence on compact sets of the form*

$$
\begin{aligned}
f_\epsilon(z) \ &= \ \tau_\epsilon z + 1 + O(1/|z|^\sigma) \\
\longrightarrow f_0(z) \ &= \ z + 1 + O(1/|z|^\sigma)
\end{aligned}
$$

for some $\sigma \in (0,1]$ *and* $\tau_\epsilon \to 1$ *α-radially. If* $R \gg 0$, *then:*

(1) For any $\epsilon \in [0,1]$ *there exists a holomorphic map* $u_\epsilon : \mathbb{V}_\epsilon(R) \to \mathbb{C}$ *such that*

$$u_\epsilon(f_\epsilon(z)) \ = \ \tau_\epsilon u_\epsilon(z) + 1.$$

(2) For any compact set K *contained in* $\mathbb{V}_\epsilon(R)$ *for all* $\epsilon \in [0,1]$, *$u_\epsilon \to u_0$ uniformly on* K.

This theorem is a mild generalization of Ueda's theorem in [Ue1] that deals with the case of $\sigma = 1$. (See also [Ue2].) This plays a crucial role to show the continuity of tessellation of the filled Julia set for hyperbolic and parabolic quadratic maps. See [Ka]. C.McMullen showed that there exist quasiconformal linearizations with much wider domain of definition. Indeed, $\tau_\epsilon \to 1$ may be tangent to the unit circle (*horocyclic* in his term). See [Mc, §8].

Remark on the domain of convergence. We can take such a compact subset K above in

$$
\begin{aligned}
\Pi(R) \ &:= \ \mathbb{C} - \left\{ e^{\theta i} z : \operatorname{Re} z < R, \ |\theta| \leq \alpha \right\} \\
&= \ \{z \in \mathbb{C} : \operatorname{Re}(z - R') \geq |z - R'| \sin\alpha\},
\end{aligned}
$$

which is a closed sector at $z = R' = R/(\cos\alpha) > 0$. In fact, for any $R > 0$ and $\epsilon \in [0,1]$, $\Pi(R)$ is contained in $\mathbb{V}_\epsilon(R)$. One can check it as follows: Now the complement of $\mathbb{V}_\epsilon(R)$ is contained in $\left\{ e^{\theta i} z : \operatorname{Re} z < R, \ \theta = \arg(-b_\epsilon) \right\}$. Since $|\arg(-b_\epsilon)| \leq \alpha$, we have the claim.

In the next section we give a proof of this theorem that is also an alternative proof of Ueda's simultaneous linearization when $\sigma = 1$. His original proof given in [Ue1] uses a technical difference equation but it makes the proof beautiful and the statement a little more detailed. Here we present a simplified proof based on the argument of [Mi, Lemma 10.10] (its idea can be traced back at least to Leau's work on the Abel equation [L]) and an estimate on polylogarithm functions given in Section 4.

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3   Proof of the theorem

Let us start with a couple of lemmas. Set $\delta := (\cos \alpha)/2 > 0$. We first check:

Lemma 3.1 *If $R \gg 0$, there exists $M > 0$ such that $|f_\epsilon(z) - (\tau_\epsilon z + 1)| \leq M/|z|^\sigma$ on $\mathbb{B}(R)$ and $N_\epsilon(f_\epsilon(z)) \geq N_\epsilon(z) + \delta$ on $\mathbb{V}_\epsilon(R)$ for any $\epsilon \in [0,1]$.*

Proof.   The first inequality and the existence of M is obvious. By replacing R by a larger one, we have $|f_\epsilon(z) - (\tau_\epsilon z + 1)| \leq M/R^\sigma < \delta$ on $\mathbb{B}(R)$. Then

$$N_\epsilon(f_\epsilon(z)) \; \geq \; N_\epsilon(\ell_\epsilon(z)) - \delta \; \geq \; N_\epsilon(z) + \delta.$$

$\blacksquare$

Let us fix such an $R \gg 0$. Then the lemma above implies that $f_\epsilon(\mathbb{V}_\epsilon(R)) \subset \mathbb{V}_\epsilon(R)$. Moreover, since $N_\epsilon(z) \leq |z|$, we have

$$|f_\epsilon^n(z)| \; \geq \; N_\epsilon(f_\epsilon^n(z)) \; \geq \; N_\epsilon(z) + n\delta \; \geq \; R + n\delta \; \to \; \infty. \tag{2.1}$$

Thus $\mathbb{V}_\epsilon(R)$ is contained in the basin of infinity and uniformly attracted to ∞ in spherical metric of $\hat{\mathbb{C}}$. In particular, this convergence to ∞ is uniform on $\Pi(R)$ for any ϵ.

Next we show a key lemma for the theorem:

Lemma 3.2 *There exists $C > 0$ such that for any $\epsilon \in [0,1]$ and z_1, $z_2 \in \mathbb{B}(2S)$ with $S > R$, we have:*

$$\left| \frac{f_\epsilon(z_2) - f_\epsilon(z_1)}{z_2 - z_1} - \tau_\epsilon \right| \; \leq \; \frac{C}{S^{1+\sigma}}.$$

Proof.   Set $g_\epsilon(z) := f_\epsilon(z) - (\tau_\epsilon z + 1)$. For any $z \in \mathbb{B}(2S)$ and $w \in \mathbb{D}(z, S) := \{w : |w - z| < S\}$, we have $|w| > S$. This implies $|g_\epsilon(w)| \leq M/|w|^\sigma < M/S^\sigma$ and thus g_ϵ maps $\mathbb{D}(z, S)$ into $\mathbb{D}(0, M/S^\sigma)$. By the Cauchy integral formula (or the Schwarz lemma), it follows that $|g_\epsilon'(z)| \leq (M/S^\sigma)/S = M/S^{1+\sigma}$ on $\mathbb{B}(2S)$.

Let $[z_1, z_2]$ denote the oriented line segment from z_1 to z_2. If $[z_1, z_2]$ is contained in $\mathbb{B}(2S)$, the inequality easily follows by

$$|g_\epsilon(z_2) - g_\epsilon(z_1)| \; = \; \left| \int_{[z_1,z_2]} g_\epsilon'(z)dz \right| \; \leq \; \int_{[z_1,z_2]} |g_\epsilon'(z)||dz| \; \leq \; \frac{M}{S^{1+\sigma}}|z_2 - z_1|$$

with $C := M$. Otherwise we have to take a roundabout way to get the estimate. Let us consider a circle with diameter $[z_1, z_2]$. Then $[z_1, z_2]$ cut the circle into two semicircles, and at least one of them is contained in $\mathbb{B}(2S)$. Let $\{z_1, z_2\}$ denote the one. Then

$$|g_\epsilon(z_2) - g_\epsilon(z_1)| \; = \; \left| \int_{\{z_1,z_2\}} g_\epsilon'(z)dz \right| \; \leq \; \int_{\{z_1,z_2\}} |g_\epsilon'(z)||dz| \; \leq \; \frac{M}{S^{1+\sigma}} \cdot \frac{\pi}{2}|z_2 - z_1|$$

and the lemma holds by setting $C := M\pi/2 \; (> M)$ for any z_1, $z_2 \in \mathbb{B}(2S)$. $\blacksquare$

Proof of Theorem 2.1. Set $z_n := f_\epsilon^n(z)$ for $z \in \mathbb{V}_\epsilon(2R)$. Note that such z_n satisfies $|z_n| \geq N_\epsilon(z_n) \geq 2R + n\delta$ by (2.1). Now we fix $a \in \mathbb{V}_\epsilon(2R)$ and define $\phi_{n,\epsilon} = \phi_n : \mathbb{V}_\epsilon(2R) \to \mathbb{C}$ $(n \geq 0)$ by

$$\phi_n(z) := \frac{z_n - a_n}{\tau_\epsilon^n}.$$

For example, one can take such an a in $\Pi(2R)$ independently of ϵ. Then we have

$$\left| \frac{\phi_{n+1}(z)}{\phi_n(z)} - 1 \right| = \left| \frac{z_{n+1} - a_{n+1}}{\tau_\epsilon(z_n - a_n)} - 1 \right| = \frac{1}{|\tau_\epsilon|} \cdot \left| \frac{f_\epsilon(z_n) - f_\epsilon(a_n)}{z_n - a_n} - \tau_\epsilon \right|.$$

We apply Lemma 3.2 with $2S = 2R + n\delta$. Since $z_n, a_n \in \mathbb{V}_\epsilon(2S) \subset \mathbb{B}(2S)$, we have

$$\left| \frac{\phi_{n+1}(z)}{\phi_n(z)} - 1 \right| \leq \frac{C}{|\tau_\epsilon|(R + n\delta/2)^{1+\sigma}} \leq \frac{C'}{(n+1)^{1+\sigma}},$$

where $C' = 2^{1+\sigma} C / \delta^{1+\sigma}$ and we may assume $R > \delta/2$. Set $P := \prod_{n \geq 1} (1 + C'/n^{1+\sigma})$. Since $|\phi_{n+1}(z)/\phi_n(z)| \leq 1 + C'/(n+1)^{1+\sigma}$, we have

$$|\phi_n(z)| = \left| \frac{\phi_n(z)}{\phi_{n-1}(z)} \right| \cdots \left| \frac{\phi_1(z)}{\phi_0(z)} \right| \cdot |\phi_0(z)| \leq P|z - a|.$$

Hence

$$|\phi_{n+1}(z) - \phi_n(z)| = \left| \frac{\phi_{n+1}(z)}{\phi_n(z)} - 1 \right| \cdot |\phi_n(z)| \leq \frac{C'P}{(n+1)^{1+\sigma}} \cdot |z - a|.$$

This implies that $\phi_\epsilon = \phi_0 + (\phi_1 - \phi_0) + \cdots = \lim \phi_n$ converges uniformly on compact subsets of $\mathbb{V}_\epsilon(2R)$ and for all $\epsilon \in [0,1]$. The univalence of ϕ_ϵ is shown in the same way as [Mi, Lemma 10.10].

Next we claim that $\phi_\epsilon(f_\epsilon(z)) = \tau_\epsilon \phi_\epsilon(z) + B_\epsilon$ with $B_\epsilon \to 1$ as $\epsilon \to 0$. One can easily check that $\phi_n(f_\epsilon(z)) = \tau_\epsilon \phi_{n+1}(z) + B_n$ where

$$B_n = \frac{a_{n+1} - a_n}{\tau_\epsilon^n} = \frac{(\tau_\epsilon - 1)a_n}{\tau_\epsilon^n} + \frac{1 + g_\epsilon(a_n)}{\tau_\epsilon^n}.$$

When $\tau_\epsilon = 1$, B_n tends to 1 since

$$|g_\epsilon(a_n)| \leq \frac{M}{|a_n|^\sigma} \leq \frac{M}{(2R + n\delta)^\sigma} \leq \frac{M}{(n\delta)^\sigma} \to 0.$$

When $|\tau_\epsilon| > 1$, the last term of the equation on B_n above tends to 0. For $n \geq 1$, we have

$$a_n = \tau_\epsilon^n a + \frac{\tau_\epsilon^n - 1}{\tau_\epsilon - 1} + \sum_{k=0}^{n-1} \tau_\epsilon^{n-1-k} g_\epsilon(a_k).$$

Thus

$$\frac{(\tau_\epsilon - 1)a_n}{\tau_\epsilon^n} = (\tau_\epsilon - 1)\left(a + \frac{g_\epsilon(a)}{\tau_\epsilon} + \sum_{k=1}^{n-1} \frac{g_\epsilon(a_k)}{\tau_\epsilon^{k+1}} \right) + 1 - \frac{1}{\tau_\epsilon^n}.$$

By the inequality on $|g_\epsilon(a_n)|$ above, we have

$$\left|(\tau_\epsilon - 1)\sum_{k=1}^{n-1}\frac{g_\epsilon(a_k)}{\tau_\epsilon^{k+1}}\right| \leq \frac{M}{\delta^\sigma}\frac{|\tau_\epsilon - 1|}{|\tau_\epsilon|}\sum_{k=1}^{n-1}\frac{1}{k^\sigma|\tau_\epsilon|^k} \leq \frac{M}{2\delta^{1+\sigma}}(1 - \frac{1}{|\tau_\epsilon|})\mathrm{Li}_\sigma(\frac{1}{|\tau_\epsilon|})$$

where we used the inequality

$$|\tau_\epsilon - 1| \leq \frac{\mathrm{Re}\,\tau_\epsilon - 1}{\cos\alpha} \leq \frac{|\tau_\epsilon| - 1}{2\delta}$$

that comes from the radial convergence. By Proposition 4.1 in the next section, B_n converges to some B_ϵ. More precisely, if we set $|\tau_\epsilon| = e^L$, then $\tau_\epsilon - 1 = O(L)$ and one can check that $B_\epsilon = 1 + O(L^{\sigma/(1+\sigma)})$.

Finally, $u_\epsilon(z) := \phi_\epsilon(z)/B_\epsilon$ gives a desired holomorphic map (with R in the statement replaced by $2R$). ∎

Remarks.

- When $\sigma = 1$, we have

$$\left|\sum_{k=1}^{n-1}\frac{g_\epsilon(a_k)}{\tau_\epsilon^{k+1}}\right| \leq \frac{M}{\delta|\tau_\epsilon|}\sum_{k=1}^{n-1}\frac{1}{k|\tau_\epsilon|^k} \leq -\frac{M}{\delta}\log(1 - \frac{1}{|\tau_\epsilon|})$$

  and this implies that $B_\epsilon = 1 + O(L|\log L|)$ if we set $|\tau_\epsilon| = e^L$. This fact is consistent with the result in [Ue1].

- By this proof, if $\{f_\epsilon(z)\}$ analytically depends on ϵ, then $\{B_\epsilon\}$ and $\{u_\epsilon(z)\}$ do the same for fixed a in $\Pi(2R)$.

- It is not difficult to check that $u_\epsilon(z) = z(B_\epsilon^{-1} + o(1))$ as $z \to \infty$ within $\mathbb{V}_\epsilon(R)$. (It is well-known that if $f_0(z) = z + 1 + a_0/z + \cdots$ then the Fatou coordinate is of the form $u_0(z) = z - a_0\log z + O(1)$. See [Sh].)

4   An estimate on polylogarithm functions

We define the *polylogarithm function* of exponent $s \in \mathbb{C}$ by

$$\mathrm{Li}_s(z) := \sum_{n=1}^\infty \frac{z^n}{n^s}.$$

This function makes sense when $|z| < 1$ and $\sigma := \mathrm{Re}\,s > 0$ and it is a holomorphic function of z. In particular, if $\mathrm{Re}\,s > 1$ the function tends to $\zeta(s)$ as $z \to 1$ within the unit disk. In the following we consider the behavior of $\mathrm{Li}_s(z)$ as $z \to 1$ within the unit disk when $0 < \sigma \leq 1$. We claim:

Proposition 4.1 *Suppose $0 < \operatorname{Re} s = \sigma \leq 1$ and $z \to 1$ with $|z| < 1$. Set $\epsilon :=$* $1 - |z|$. *Then there exists a uniform constant C independent of s such that*

$$|\operatorname{Li}_s(z)| \leq C\epsilon^{-\frac{1}{1+\sigma}}$$

as $z \to 1$. In particular, we have

$$|(z-1)\operatorname{Li}_s(z)| \leq C\epsilon^{\frac{\sigma}{1+\sigma}} \to 0$$

if $z \to 1 - 0$ along the real axis.

Proof. Clearly $|\operatorname{Li}_s(z)| \leq \sum_{n=1}^{\infty} |z|^n/n^\sigma$ so it is enough to consider the sum

$$S := \sum_{n=1}^{\infty} \frac{1}{n^\sigma} \cdot \lambda^n$$

where $\lambda := |z| = 1 - \epsilon$. Let S_n be the partial sum to the nth term. By the Hölder inequality, we have

$$S_n \leq \left(\sum_{k=1}^{n} \frac{1}{k^{\sigma p}} \right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} \lambda^{kq} \right)^{\frac{1}{q}}$$

for any $p, q > 1$ with $1/p + 1/q = 1$. Now let us set $p := 1/\sigma + 1 \geq 2$ (then $1 < q = 1 + \sigma \leq 2$). Since $\sigma p = 1 + \sigma > 1$, the first sum is uniformly bounded as follows:

$$\sum_{k=1}^{n} \frac{1}{k^{\sigma p}} \leq 1 + \int_1^\infty \frac{1}{x^{1+\sigma}} dx = 1 + \frac{1}{\sigma} = p.$$

On the other hand, for the second sum, we still have $0 < \lambda^q < 1$ and thus

$$\sum_{k=1}^{n} \lambda^{kq} \leq \frac{\lambda^q}{1 - \lambda^q} = \frac{1}{q\epsilon}(1 + o(1)) \leq \frac{2}{q\epsilon}$$

when $\epsilon \ll 1$. Hence we have the following uniform bound:

$$S_n \leq p^{\frac{1}{p}} \left(\frac{2}{q\epsilon} \right)^{\frac{1}{q}} \leq 2 \left(\frac{p^{\frac{1}{p}}}{q^{\frac{1}{q}}} \right) \epsilon^{-\frac{1}{q}}.$$

One can easily check that $1 \leq x^{\frac{1}{x}} \leq e^{\frac{1}{e}} = 1.44467 \cdots$ for $x \geq 1$. Thus

$$S \leq 2 e^{\frac{1}{e}} \epsilon^{-\frac{1}{q}} = 2 e^{\frac{1}{e}} \epsilon^{-\frac{1}{1+\sigma}}$$

when $\epsilon \ll 1$ and we have the desired estimate with $C = 2e^{\frac{1}{e}} < 3$. The last inequality of the statement follows by:

$$|(z-1)\operatorname{Li}_s(z)| \leq C\epsilon^{1-\frac{1}{q}} = C\epsilon^{\frac{1}{p}} = C\epsilon^{\frac{\sigma}{1+\sigma}}.$$

(Indeed, $|(z-1)\operatorname{Li}_s(z)| = O(\epsilon^{\frac{\sigma}{1+\sigma}})$ if $z \to 1$ radially.) ∎

5 Application: Proof of Theorem 1.2

As an application of Theorem 2.1, we give a proof of Theorem 1.2. Though Theorem 1.2 only deals with the simplest parabolic fixed point and its simplest perturbation, one can easily extend the result to general parabolic cycles with multiple petals and their "non-tangential" perturbations.

Proof of Theorem 1.2. Let us take an general expression $f_\lambda(w) = \lambda w + w^2$ with $0 < \lambda \leq 1$ (thus $f_1 = g$). By looking the action of f_λ through a new coordinate $z = \chi_\lambda(w) = -\lambda^2/w$, we have

$$\chi_\lambda \circ f_\lambda \circ \chi_\lambda^{-1}(z) \;=\; z/\lambda + 1 + O(1/z)$$

near ∞. Now we can set $\tau_\epsilon := 1/\lambda = 1 + \epsilon$ and $f_\epsilon := \chi_\lambda \circ f_\lambda \circ \chi_\lambda^{-1}$ to have the same setting as Theorem 2.1. We consider that f and g are parameterized by λ or ϵ. (It is convenient to use both parameterization.) Note that $\Pi(R) = \{\operatorname{Re} z \geq R\}$ in this case. By the same argument as Lemma 3.1, we can check that $\operatorname{Re} f_\epsilon(z) \geq \operatorname{Re} z + 1/2$ if $z \in \Pi(R)$ and $R \gg 0$. In particular, we have $f_\epsilon(\Pi(R)) \subset \Pi(R)$ for $R \gg 0$.

Let us show (1): For any compact $E \subset K_g^\circ$ and small $r > 0$, there exists $N \gg 0$ such that $g^N(E) \subset P_r = \{|w + r| \leq r\}$. (For instance, one can show this fact by existence of the Fatou coordinate.) By uniform convergence, we have $f^N(E) \subset P_r$ for all $f \approx g$. To show $E \subset K_f^\circ$, it is enough to show that $f(P_r) \subset P_r$ for all $f \approx g$. Since $\chi_\lambda(P_r) = \Pi(R)$ for some $R \gg 0$, we have $f_\epsilon(\Pi(R)) \subset \Pi(R)$ independently of ϵ. This is equivalent to $f_\lambda(P_r) \subset P_r$ in a different coordinate. Thus we have (1).

Next let us check (2): Set $\Phi_\epsilon := \Phi_f$ and $\Phi_0 := \Phi_g$. Then we have $\Phi_\epsilon(f_\lambda(w)) = \tau_\epsilon \Phi_\epsilon(w) + 1$. On the other hand, by simultaneous linearization, we have a uniform convergence $u_\epsilon \to u_0$ on $\Pi(R)$ that satisfies $u_\epsilon(f_\epsilon(z)) = \tau_\epsilon u_\epsilon(z) + 1$. By setting $\Psi_\epsilon(w) := u_\epsilon \circ \chi_\lambda(w)$, we have $\Psi_\epsilon \to \Psi_0$ compact uniformly on P_r, and $\Psi_\epsilon(f_\lambda(w)) = \tau_\epsilon \Psi_\epsilon(w) + 1$.

We need to adjust the images of critical orbits mapped by Φ_ϵ and Ψ_ϵ. Since $g^n(-1/2) \to 0$ along the real axis, there is an $M \gg 0$ such that $g^M(-1/2) =: a_0 \in P_r$. By uniform convergence, we also have $f^M(-\lambda/2) =: a_\epsilon \in P_r$ and $a_\epsilon \to a_0$ as $\epsilon \to 0$. Set $b_\epsilon := \Psi_\epsilon(a_\epsilon)$ and $c_\epsilon := \Phi_\epsilon(a_\epsilon)$ for all $\epsilon \geq 0$. Set also $\ell_\epsilon(W) = \tau_\epsilon W + 1$, then we have $c_\epsilon = \ell_\epsilon^M(0) = \tau_\epsilon^{M-1} + \cdots + \tau_\epsilon + 1$ and $c_\epsilon \to c_0 = M$ as $\epsilon \to 0$. When $\epsilon > 0$, we take an affine map T_ϵ that fixes $1/(1 - \tau_\epsilon)$ and sends b_ϵ to c_ϵ. When $\epsilon = 0$, we take T_0 that is the translation by $b_0 - c_0$. Then one can check that $T_\epsilon \to T_0$ compact uniformly on the plane and $\tilde{\Phi}_\epsilon := T_\epsilon \circ \Psi_\epsilon$ satisfies $\tilde{\Phi}_\epsilon \to \tilde{\Phi}_0$ on any compact sets of P_r. Moreover, $\tilde{\Phi}_\epsilon$ still satisfies $\tilde{\Phi}_\epsilon(f_\lambda(w)) = \tau_\epsilon \tilde{\Phi}_\epsilon(w) + 1$ and the images of the critical orbit by Φ_ϵ and $\tilde{\Phi}_\epsilon$ agree. Finally by uniqueness of ϕ_f and ϕ_g, one can check that $\Phi_\epsilon = \tilde{\Phi}_\epsilon$ on P_r.

Since

$$\Phi_f(w) \;=\; \ell_\epsilon^{-N} \circ \tilde{\Phi}_\epsilon(f^N(w)) \;\longrightarrow\; \ell_0^{-N} \circ \tilde{\Phi}_0(g^N(w)) \;=\; \Phi_g(w)$$

uniformly on E, we have (2). ∎

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